

Poincaré's topology at the interplay between syntax and semantics

Mark Wilson proposes an original account of the grip our linguistic tools give us on the world, as illustrated in science and in mathematics. By studying for instance the development of projective geometry, Wilson describes a conflict between what he calls an apparent and an active grammar, reflecting the seemingly contradictory requirements of a powerful syntax and a coherent semantics. Wilson's originality lies in his refusal to see this duality as a transitory aspect of science, to be dismissed by the adoption of a proper and definitive setting ; rather, he emphasizes the necessity of long periods of « semantic agnosticism » in the emergence of certain mathematical fields, in order to be able to reap the inferential benefits of new syntactic tools, long before engaging in a systematic semantic clarification of this particular domain ; what is more, this clarification is never assured to be definitive ; in fact, we have to consider this opposition as a productive dialectic that is pervasive in human conceptual behavior.

We would like to study some parts of Poincaré's work on topology (a field he contributed to create to an essential degree) according to Wilson's lens. But instead of focusing on the syntactic requirement by which Wilson characterizes the emergence of a new discipline, we think that Poincaré is preoccupied in turn with both the syntactical and the semantical requirements, so that his « Analysis situs » exhibits in a compressed manner the dialectic Wilson describes in the longer run. In the introduction of his first memoir, Poincaré expresses a semantical concern ; he conforms itself to a tradition going back to Poncelet by making the case for a geometrical language, better equipped to give us comprehension as an analytical one. Nevertheless, as far as language is concerned, Poincaré's main innovation is algebraic ; a fact exemplified for instance with the introduction of the homologies, which gives us a powerful syntactic tool, by simplifying steps necessary to define topological invariants. But as the same time it exerts a shift in the geometrical interpretation of topological objects.

The most dramatic evidence of a tension between an apparent and an active grammar is then to be found in the conflict between Poincaré and Heegard about the duality theorem. At the root of the opposition between the two mathematicians lies a disagreement about the range of the operations permitted by the new formalism. In order to answer Heegard's critique and save the validity of this essential piece of topology, Poincaré is compelled, in a reply to Heegard and then in the first complement to the « Analysis situs », to extend the power of his algebraic tool, once more at the cost of the geometrical interpretation ; but he was not satisfied with this loss in semantics and strived to provide a new one ; which he did soon afterwards in the second complement , giving rise to the new and fruitful notion of torsion.

According to Poincaré, topology was a central piece of mathematical knowledge, a fact he explained by its exhibiting geometrical intuition in its purity ; we think that it can be explained, thanks to Wilson's conceptuality, in a more complex manner, which can in turn explain Weyl's opinion according to which « the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics. »