Two ways to think about (implicit) structure

According to a dominant view in modern philosophy of mathematics, mathematics can be understood as the study of abstract structures (e.g. [2], [8]). Put differently, structuralism holds that theories of pure mathematics (such as Peano arithmetic, lattice theory, topology, or graph theory) study only the structure or the structural properties of their respective subject fields (namely number systems, lattices, topological spaces, and graphs). But what precisely is the relevant structure of such mathematical entities? How can we think about their structural content? The present talk will address these questions based on a distinction between *primitive* and *implicit* structure, that is, between systems defined by mathematical theories and their implicit structural content.

As is well-known, mathematical objects such as lattices or topological spaces are usually introduced axiomatically today, that is, in terms of formal axiomatic definitions that specify the constitutive properties of the objects in question. For instance, a topological space can be defined in terms so-called neighborhood axioms, first introduced by Felix Hausdorff in 1914, which specify the properties of a neighborhood relation between points and sets of point sets. While it is clear that the primitive properties of mathematical systems are specified axiomatically in this sense, less has been said about how mathematicians investigate the *implicit* structure of such systems. In this talk, we will compare two general ways to think about this implicit structural content of theories of pure mathematics. According to the first approach, the implicit structure or the structural properties of mathematical objects are specified with reference to formal languages, usually based on some notion of definability. Thus, properties of systems such as rings or graphs will count as structural if they are logically definable in a formal language of the correct mathematical signature. According to the second approach, structures are determined in terms of invariance criteria. For instance, the structural properties of a given mathematical system are often said to be those properties invariant under certain transformations of the system or under mappings between similar systems (see [1], [6]). In the talk, we will investigate these two approaches by drawing to a particular mathematical case study, namely the study of simple incidence structures in *finite* affine and projective geometry.

Given these geometrical examples, we give a philosophical analysis of the conceptual differences between the two methods to express implicit structure. The talk will focus on three issues. The first concerns the conceptual motivation for treating mathematical structures in terms of the notions of definability and invariance. Why are these criteria adequate means for the specification of structural properties? As will be argued, both methods capture some form of "topic neutrality" underlying the structuralist account of mathematics. In the case of invariance, this is due to the fact that mathematics is indifferent to the intrinsic nature of mathematical objects and thus also indifferent to arbitrary switchings of such objects in a given system. In the case of definability-based approaches, the relevant topic neutrality is related to the fact that adequate logical definitions should be reducible to statements about the primitive mathematical structure. The structural topic neutrality in mathematics thus seems to be explainable in terms of the “formality” of logic (see [3]).

The second point addressed in the talk concerns the logical relation between two approaches to implicit structure. As work in logic and model theory has shown, the exist a general *symmetry* or *duality* between the method of specifying invariants relative to transformations or mappings and the notion of definability (see, in particular, [5]). We will present a formal account of this duality in terms of a Galois connection between automorphism classes and Galois-closed sets of relations. More specifically, given this framework, it will be shown that the class of definable properties (of the objects) of a given primitive structure always forms a subclass of the class of properties invariant under the automorphisms of the structure.

Finally, we discuss the relevance of the two ways to think about implicit structure for our understanding of mathematical structuralism. Here, in particular, the focus will be on the notion of the *equivalence* of mathematical structures (see [7], [8]). Building on the existing literature on the topic, we will propose two notions of structural equivalence that take into account not only the (axiomatically defined) primitive structure, but also its implicit structural content. The first notion is motivated by the idea of definable implicit structure and based on the notion of *interpretability*. According to it, two mathematical structures are equivalent if they are bi-interpretable (compare [4]). The second notion, in turn, is motivated by the invariant approach and based on the concept of ”transfer principles” between structures. According to it, two structures are equivalent if there exists a mapping between their domains that induces an isomorphism of the respective automorphism groups.

References

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